Genericity of Caustics on a corner

Takaharu Tsukada[1]

College of Humanities & Sciences, Department of Mathematics,
Nihon University

Abstract

We introduce the notions of *the caustic-equivalence* and *the weak caustic-equivalence relations* of reticular Lagrangian maps in order to give a generic classification of caustics on a corner. We give the figures of all generic caustics on a corner in a smooth manifold of dimension 2 and 3.

1 Introduction

In [**?**] we investigate the theory of *reticular Lagrangian maps* which can be described stable caustics generated by a hypersurface germ with an r-corner in a smooth manifold. A map germ

$$\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

is called *a reticular Lagrangian map* if i is a restriction of a symplectic diffeomorphism germ on $(T^*\mathbb{R}^n, 0)$, where $I_r = \{1, \ldots, r\}$ and $\mathbb{L} = \{(q,p) \in T^*\mathbb{R}^n | q_1p_1 = \cdots = q_rp_r = q_{r+1} = \cdots = q_n = 0, q_{I_r} \geq 0\}$ be a representative of the union of

$$L^0_\sigma = \{(q,p) \in (T^*\mathbb{R}^n, 0) | q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = 0, q_{I_r - \sigma} \geq 0\} \text{ for all } \sigma \subset I_r.$$

We define the caustic of $\pi \circ i$ is the union of the caustics C_σ of the Lagrangian maps $\pi \circ i|_{L^0_\sigma}$ for all $\sigma \subset I_r$ and the quasi-caustic $Q_{\sigma,\tau} = \pi \circ i(L^0_\sigma \cap L^0_\tau)$ for all $\sigma, \tau \subset I_r (\sigma \neq \tau)$. In the case $r = 2$, that is the initial hypersurface germ has a corner, the caustic of $\pi \circ i$ is

$$C_\emptyset \cup C_1 \cup C_2 \cup C_{1,2} \cup Q_{\emptyset,1} \cup Q_{\emptyset,2} \cup Q_{1,\{1,2\}} \cup Q_{2,\{1,2\}}.$$

For the definitions of generating families of reticular Lagrangian maps, see [**?**, p.575-577]. In [**?**] we investigate the genericity of caustics on an r-corner and give the generic classification for the cases $r = 0$ and 1 by using G.Ishikawa's methods (see [**?**, Section 5]). We also showed that this method do not work well for the case $r = 2$. In this paper we introduce the two equivalence relations of reticular Lagrangian maps which are weaker than Lagrangian equivalence in order to give a generic classification of caustics on a corner.

2 Caustic-equivalence and Weak caustic-equivalence

We introduce the equivalence relations of reticular Lagrangian maps and their generating families.

Let $\pi \circ i_j$ be reticular Lagrangian maps for $j = 1, 2$. We say that they are *caustic-equivalent* if there exists a diffeomorphism germ g on $(\mathbb{R}^n, 0)$ such that

$$g(C^1_\sigma) = C^2_\sigma, \quad g(Q^1_{\sigma,\tau}) = Q^2_{\sigma,\tau} \quad \text{for all } \sigma, \tau \subset I_r \ (\sigma \neq \tau). \tag{1}$$

[1]Higashijujo 3-1-16 Kita-ku, Tokyo 114-0001 JAPAN. e-mail : tsukada@math.chs.nihon-u.ac.jp

Example of stable caustics on a corner

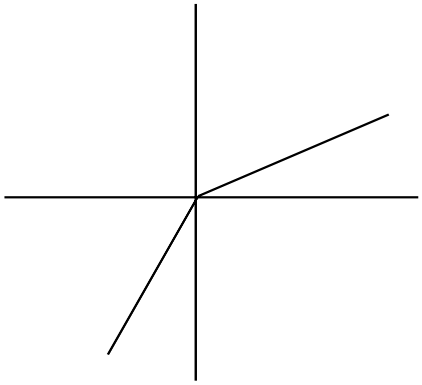

Figure 1: weakly caustic-stable

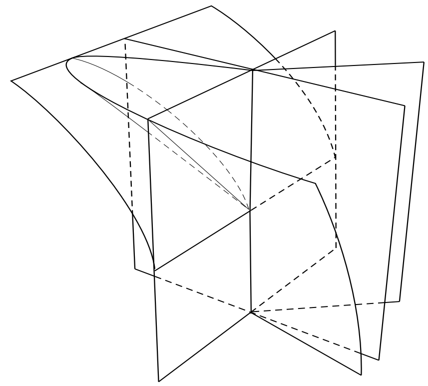

Figure 2: caustic-stable

We say that reticular Lagrangian maps $\pi \circ i_1$ and $\pi \circ i_2$ are *weakly caustic-equivalent* if there exists a homeomorphism germ g on $(\mathbb{R}^n, 0)$ such that g is smooth on all C^1_σ, $Q^1_{\sigma,\tau}$, and satisfies (1).

We shall define the stabilities of reticular Lagrangian maps under the above equivalence relations and define the corresponding equivalence relations and stabilities of their generating families.

The purpose of this paper is to show the following theorem:

Theorem 2.1 *Let $n = 2, 3$ or 4, U a neighborhood of 0 in $T^*\mathbb{R}^n$, $S(T^*\mathbb{R}^n, 0)$ be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$, and $S(U, T^*\mathbb{R}^n)$ be the space of symplectic embeddings from U to $T^*\mathbb{R}^n$ with C^∞-topology. Then there exists a residual set $O \subset S(U, T^*\mathbb{R}^n)$ such that for any $\tilde{S} \in O$ and $x \in U$, the reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ is weakly caustic-stable or caustic-stable, where $\tilde{S}_x \in S(T^*\mathbb{R}^n, 0)$ be defined by the map $x_0 \mapsto \tilde{S}(x_0 + x) - \tilde{S}(x)$.*

A reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ for any $\tilde{S} \in O$ and $x \in U$ is weakly caustic-equivalent to one which has a generating family $B^{\pm,+,1}_{2,2}, B^{\pm,+,2}_{2,2}, B^{\pm,-}_{2,2}$, or is caustic equivalent to one which has a generating family $B^{\pm,0}_{2,2}, B^{\pm.\pm}_{2,2,3}, B^{\pm,\pm}_{2,3}, B^{\pm,\pm}_{3,2}, B^{\pm,\pm}_{2,3'}, B^{\pm,\pm}_{3,2'}, C^{\pm,\pm}_{2,3}, C^{\pm,\pm}_{3,2,1}, C^{\pm,\pm}_{3,2,2}$.
$B^{\pm,+,1}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,+,2}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,-}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 - x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,0}_{2,2}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1x_2$,
$B^{\pm,\pm}_{2,2,3}$: $F(x_1, x_2, q_1, q_2, q_3) = (x_1 \pm x_2)^2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,
$B^{\pm,\pm}_{2,3}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 \pm x_1x_2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,
$B^{\pm,\pm}_{3,2}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^3 \pm x_1x_2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1^2$,
$B^{\pm,\pm}_{2,3'}$: $F(x_1, x_2, q_1, q_2, q_3, q_4) = x_1^2 \pm x_1x_2^2 \pm x_2^3 + q_1x_2^2 + q_2x_1x_2 + q_3x_2 + q_4x_1$,
$B^{\pm,\pm}_{3,2'}$: $F(x_1, x_2, q_1, q_2, q_3, q_4) = x_1^3 \pm x_1^2x_2 \pm x_2^2 + q_1x_1^2 + q_2x_1x_2 + q_3x_1 + q_4x_2$,
$C^{\pm,\pm}_{3,2}$: $F(y, x_1, x_2, q_1, q_2, q_3) = \pm y^3 + x_1y \pm x_2y + x_2^2 + q_1y + q_2x_1 + q_3x_2$,
$C^{\pm,\pm}_{3,2,1}$: $F(y, x_1, x_2, q_1, q_2, q_3, q_4) = \pm y^3 + x_1y \pm x_2y^2 + x_2^2 + q_1y^2 + q_2y + q_3x_1x_2 + q_4x_2$,

$C^{\pm,\pm}_{3,2,2}$: $F(y,x_1,x_2,q_1,q_2,q_3,q_4)=\pm y^3+x_2y\pm x_1y^2+x_1^2+q_1y^2+q_2y+q_3x_1x_2+q_4x_1$.

In order to describe the caustic-equivalence of reticular Lagrangian maps by their generating families, we introduce the following equivalence relation of function germs. We say that function germs $f,g\in\mathcal{E}(r;k)$ are *reticular $\mathcal{C}$-equivalent* if there exist $\phi\in\mathcal{B}(r;k)$ and a non-zero number $a\in\mathbb{R}$ such that $g=a\cdot f\circ\phi$. See [?] or [?] for the notations. We construct the theory of unfoldings with respect to the corresponding equivalence relation. Then the relation of unfoldings is given as follows: Two function germs $F(x,y,q),G(x,y,q)\in\mathcal{E}(r;k+n)$ are *reticular $\mathcal{P}$-$\mathcal{C}$-equivalent* if there exist $\Phi\in\mathcal{B}_n(r;k+n)$ and a unit $a\in\mathcal{E}(n)$ and $b\in\mathcal{E}(n)$ and such that $G=a\cdot F\circ\Phi+b$. We define the *stable* reticular ($\mathcal{P}$-)$\mathcal{C}$-equivalence by the ordinary ways (see [?, p.576]). We remark that a reticular $\mathcal{P}$-$\mathcal{C}$-equivalence class includes the reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalence classes.

We review the results of the theory. Let $F(x,y,u)\in\mathfrak{M}(r;k+n)$ be an unfolding of $f(x,y)\in\mathfrak{M}(r;k)$.

We say that F is *reticular $\mathcal{P}$-$\mathcal{C}$-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F}\in C^\infty(U,\mathbb{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ in C^∞-topology such that for any element $\tilde{G}\in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r\times\mathbb{R}^{k+n}}$ at $(0,y_0,q_0)$ is reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to F for some $(0,y_0,q_0)\in U$.

We say that F is *reticular $\mathcal{P}$-$\mathcal{C}$-versal* if all unfolding of f is reticular $\mathcal{P}$-$\mathcal{C}$-f-induced from F. That is, for any unfolding $G\in\mathfrak{M}(r;k+n')$ of f, there exist $\Phi\in\mathfrak{M}(r;k+n',r;k+n)$ and a unit $a\in\mathcal{E}(n')$ and $b\in\mathcal{E}(n')$ satisfying the following conditions:
(1) $\Phi(x,y,0)=(x,y,0)$ for all $(x,y)\in(\mathbb{H}^r\times\mathbb{R}^k,0)$ and $a(0)=1,\ b(0)=0$,
(2) Φ can be written in the form:

$$\Phi(x,y,q)=(x_1\phi_1^1(x,y,q),\cdots,x_r\phi_1^r(x,y,q),\phi_2(x,y,q),\phi_3(q)),$$

(3) $G(x,y,q)=a(q)\cdot F\circ\Phi(x,y,q)+b(q)$ for all $(x,y,q)\in(\mathbb{H}^r\times\mathbb{R}^{k+n'},0)$.

We say that F is *reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally versal* if

$$\mathcal{E}(r;k)=\langle x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)}+\langle 1,f,\frac{\partial F}{\partial q}|_{q=0}\rangle_{\mathbb{R}}.$$

We say that F is *reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally stable* if

$$\mathcal{E}(r;k+n)=\langle x\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\rangle_{\mathcal{E}(r;k+n)}+\langle 1,F,\frac{\partial F}{\partial q}\rangle_{\mathcal{E}(n)}.$$

We say that F is *reticular $\mathcal{P}$-$\mathcal{C}$-homotopically stable* if for any smooth path-germ $(\mathbb{R},0)\to\mathcal{E}(r;k+n),t\mapsto F_t$ with $F_0=F$, there exists a smooth path-germ $(\mathbb{R},0)\to\mathcal{B}_n(r;k+n)\times\mathcal{E}(n)\times\mathcal{E}(n),t\mapsto(\Phi_t,a_t,b_t)$ with $(\Phi_0,a_0,b_0)=(id,1,0)$ such that each (Φ_t,a_t,b_t) is a reticular $\mathcal{P}$-$\mathcal{C}$-isomorphism from F to F_t, that is $F_t=a_t\cdot F\circ\Phi_t+b_t$ for t around 0.

Theorem 2.2 (cf., [?, Theorem 4.5]) *Let $F\in\mathfrak{M}(r;k+n)$ be an unfolding of $f\in\mathfrak{M}(r;k)$. Then the following are all equivalent.*
(1) *F is reticular $\mathcal{P}$-$\mathcal{C}$-stable.*

(2) *F is reticular $\mathcal{P}$-$\mathcal{C}$-versal.*
(3) *F is reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally versal.*
(4) *F is reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally stable.*
(5) *F is reticular $\mathcal{P}$-$\mathcal{C}$-homotopically stable.*

For a non-quasihomogeneous function germ $f(x,y) \in \mathfrak{M}(r;k)$, if $1, f, a_1, \ldots, a_n \in \mathcal{E}(r;k)$ is a representative of a basis of the vector space

$$\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)},$$

then the function germ $f + a_1 q_1 + \cdots + a_n q_n \in \mathfrak{M}(r;k+n)$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of f. We call n the reticular $\mathcal{C}$-codimension of f. If f is quasihomogeneous then f is included in $\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)}$. This means that the reticular $\mathcal{C}$-codimension of a quasihomogeneous function germ is equal to its reticular $\mathcal{R}^+$-codimension.

We define the *simplicity* of function germs under the reticular $\mathcal{C}$-equivalence in the usual way (cf., [?]).

Theorem 2.3 (cf., [?, Theorem 2.1,2.2]) *A reticular $\mathcal{C}$-simple function germ in $\mathfrak{M}(1;k)^2$ is stably reticular $\mathcal{C}$-equivalent to one of the following function germs:*

$$B_l : x^l \ (l \geq 2), \quad C_l^{\varepsilon} : xy + \varepsilon y^l \ (\varepsilon^{l-1} = 1, l \geq 3), \quad F_4 : x^2 + y^3.$$

The relation between reticular Lagrangian maps and their generating families under the caustic-equivalence are given as follows:

Proposition 2.4 *Let $\pi \circ i_j$ be reticular Lagrangian maps with generating families F_j for $j = 1, 2$. If F_1 and F_2 are stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent then $\pi \circ i_1$ and $\pi \circ i_2$ are caustic-equivalent.*

Proof. The function germ F_2 may be written that $F_2(x,y,q) = a(q)F_3(x,y,q)$, where a is a unit and F_1 and F_3 are stably reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent. Then the reticular Lagrangian map $\pi \circ i_3$ given by F_3 and $\pi \circ i_1$ are Lagrangian equivalent and the caustic of $\pi \circ i_2$ and $\pi \circ i_3$ coincide to each other. ∎

This proposition shows that it is enough to classify function germs under the stable reticular $\mathcal{P}$-$\mathcal{C}$-equivalence in order to classify reticular Lagrangian maps under the caustic-equivalence. We here give the classification list as the following:

Theorem 2.5 (cf., [?, p.592]) *Let $f \in \mathfrak{M}(2;k)^2$ have the reticular $\mathcal{C}$-codimension≤ 4. Then f is stably reticular $\mathcal{C}$-equivalent to one of the following list.*

k	*Normal form*	*codim*	*Conditions*	*Notation*
0	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$0 < a < \frac{1}{4}$	$B_{2,2,a}^{\pm,+,1}$
	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$a > \frac{1}{4}$	$B_{2,2,a}^{\pm,+,2}$
	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$a < 0$	$B_{2,2,a}^{\pm,-}$
	$x_1^2 \pm x_2^2$	3		$B_{2,2}^{\pm,0}$
	$(x_1 \pm x_2)^2 \pm x_2^3$	3		$B_{2,2,3}^{\pm,\pm}$
	$x_1^2 \pm x_1x_2 \pm x_2^3$	3		$B_{2,3}^{\pm,\pm}$
	$x_1^3 \pm x_1x_2 \pm x_2^2$	3		$B_{3,2}^{\pm,\pm}$
	$x_1^2 \pm x_1x_2^2 \pm x_2^3$	4		$B_{2,3'}^{\pm,\pm}$
	$x_1^3 \pm x_1^2x_2 \pm x_2^2$	4		$B_{3,2'}^{\pm,\pm}$
1	$\pm y^3 + x_1y \pm x_2y + x_2^2$	3		$C_{3,2}^{\pm,\pm}$
	$\pm y^3 + x_1y \pm x_2y^2 + x_2^2$	4		$C_{3,2,1}^{\pm,\pm}$
	$\pm y^3 + x_2y \pm x_1y^2 + x_1^2$	4		$C_{3,2,2}^{\pm,\pm}$

We remark that the stable reticular $\mathcal{C}$-equivalence class $B_{2,3}^{+,+}$ of $x_1^2+x_1x_2+x_2^3$ consists of the union of the stable reticular $\mathcal{R}$-equivalence classes of $x_1^2+x_1x_2+ax_2^3$ and $-x_1^2-x_1x_2-ax_2^3$ for $a>0$. The same things hold for $B_{2,2,3}^{\pm,\pm}$, $B_{2,3}^{\pm,\pm}$, $B_{3,2}^{\pm,\pm}$, $C_{3,2}^{\pm,\pm}$.

3 Caustic-stability

We define *the caustic-stability* of reticular Lagrangian maps and reduce our investigation to finite dimensional jet spaces of symplectic diffeomorphism germs.

We say that a reticular Lagrangian map $\pi\circ i$ is *caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of i and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of S, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at x_0 and $\pi\circ i$ are caustic-equivalent for some $x_0 = (0,\ldots,0,p_{r+1}^0,\ldots,p_n^0) \in U$.

Definition 3.1 *Let $\pi\circ i$ be a reticular Lagrangian map and l be a non-negative number. We say that $\pi\circ i$ is* caustic l-determined *if the following condition holds: For any extension S of i, the reticular Lagrangian map $\pi\circ S'|_{\mathbb{L}}$ and $\pi\circ i$ are caustic-equivalent for any symplectic diffeomorphism germ S' on $(T^*\mathbb{R}^n,0)$ satisfying $j^lS(0) = j^lS'(0)$.*

Lemma 3.2 *Let $\pi\circ i : (\mathbb{L},0) \to (T^*\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a reticular Lagrangian map. If a generating family of $\pi\circ i$ is reticular $\mathcal{P}$-$\mathcal{C}$-stable then $\pi\circ i$ is caustic $(n+2)$-determined.*

Proof. This is proved by the analogous method of [**?**, Theorem 5.3]. We give the sketch of proof. Let S be an extension of i. Then we may assume that there exists a function germ $H(Q,p)$ such that the canonical relation P_S has the form:

$$P_S = \{(Q, -\frac{\partial H}{\partial Q}(Q,p), -\frac{\partial H}{\partial p}(Q,p), p) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, (0,0))\}.$$

Then the function germ $F(x,y,q) = H_0(x,y) + \langle y,q\rangle$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable generating family of $\pi\circ i$, and H_0 is reticular $\mathcal{R}$-$(n+3)$-determined, where $H_0(x,y) = H(x,0,y)$. Let a symplectic diffeomorphism germ S' on $(T^*\mathbb{R}^n,0)$ satisfying $j^{n+2}S(0) = j^{n+2}S'(0)$ be given. Then there exists a function germ $H'(Q,p)$ such that the canonical relation $P_{S'}$ is given the same form for H' and the function germ $G(x,y,q) = H_0'(x,y) + \langle y,q\rangle$ is a generating family

of $\pi \circ S'|_{\mathbb{L}}$. Then it holds that $j^{n+3}H_0(0) = j^{n+3}H_0'(0)$. There exists a function germ G' such that G and G' are reticular $\mathcal{P}$-$\mathcal{R}$-equivalent and F and G' are reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimal versal unfoldings of $H_0(x, y)$. It follows that F and G are reticular $\mathcal{P}$-$\mathcal{C}$-equivalent. Therefore $\pi \circ i$ and $\pi \circ S'|_{\mathbb{L}}$ are caustic-equivalent. ■

For a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding $F \in \mathfrak{M}(2; k+n)^2$ with $n \leq 3$, the function germ $f = F|_{q=0}$ has a modality under the reticular $\mathcal{R}$-equivalence (see [?, p.592]). For example, consider the case f is stably reticular $\mathcal{C}$-equivalent to $x_1^2 + x_1x_2 + x_2^3$. Then F is stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to $f + q_1x_1 + q_2x_2 + q_3x_2^2$. In this case the function germs $F_a(x, q) = x_1^2 + x_1x_2 + ax_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2 (a > 0)$ are stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to F but not stably reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to each other. Let $S_a^{\pm}$ be extensions of reticular Lagrangian embeddings defined by F_a and $-F_a$ for $a > 0$ respectively. We define the caustic-equivalence class of S_1 by

$$[S_1]_c := \bigcup_{a>0}([S_a^+]_L \cup [S_a^-]_L),$$

where $[S_a^{\pm}]_L$ are the Lagrangian equivalence classes of $S_a^{\pm}$ respectively. By Proposition 2.4, we have that all reticular Lagrangian maps $\pi \circ S'|_{\mathbb{L}}$ are caustic-equivalent to each other for $S' \in [S_1]_c$.

In order to apply the transversality theorem to our theory, we need to prove that the set consists of the 2-jets of the caustic-equivalence class $[S_1]_c$, we denote this by $[j^2S_1(0)]_c$, is an immersed manifold of $S^2(3)$, where $S^l(n)$ be the smooth manifold which consists of l-jets of elements in $S(T^*\mathbb{R}^n, 0)$. We shall prove that the map germ $(0, \infty) \to S^2(3), a \mapsto j^2S_a(0)$ is not tangent to $[j^2S_a(0)]_L$ for any a, and apply the following lemma:

Lemma 3.3 *Let I be an open interval, N a manifold, and G a Lie group acts on N smoothly. Suppose the orbits $G \cdot x$ have the same dimension for all $x \in I$. Let $x : I \to N$ be a smooth path such that $\frac{dx}{dt}(t)$ is not tangent to $G \cdot x(t)$ for all $t \in I$. Then*

$$\bigcup_{t \in I} G \cdot x(t)$$

is an immersed manifold of N.

We denote that we here prove the case $B_{2,3}^{+,+}$. The same method is valid for all $B_{2,3}^{\pm,\pm}, B_{3,2}^{\pm,\pm}$.

We define $G_a \in \mathfrak{M}(6)^2$ by $G_a(Q_1, Q_2, Q_3, q_1, q_2, q_3) = F_a(Q_1, Q_2, q_1, q_2) + Q_3q_3$. Then G_a define the canonical relations P_a and they give symplectic diffeomorphisms S_a of the forms:

$$S_a(Q, P) = (-2Q_1 - Q_2 - P_1, -Q_1 - 3aQ_2^2 - P_2 + 2P_3Q_2, -P_3, Q_1, Q_2, Q_2^2 + Q_3).$$

We have that F_a are generating families of $\pi \circ S_a|_{\mathbb{L}}$. Then $\frac{dS_a}{da} = (0, -3Q_2^2, 0, 0, 0, 0) = X_f \circ S_a$ for $f = -p_2^3$. We suppose that $j^2(\frac{dS_a}{da})(0) \in T_z([z]_L)$ for $z = j^2S_a(0)$. By [?, Lemma 6.2], there exist a fiber preserving function germ $H \in \mathfrak{M}_{Q,P}^2$ and $g \in \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle$ such that $j^2(X_f \circ S_a)(0) = j^2(X_H \circ S_a + (S_a)_*X_g)(0)$. This means that $j^3(f \circ S_a)(0) = j^3(H \circ S_a + g)(0)$. It follows that there exist function germs $h_1, h_2, h_3 \in \mathfrak{M}_Q$, $h_0 \in \mathfrak{M}_Q^2$ such that

$$f \circ S_a = -Q_2^3 \equiv h_1(q \circ S_a)Q_1 + h_2(q \circ S_a)Q_2 + h_3(q \circ S_a)(Q_2^2 + Q_3) + h_0(q \circ S_a)$$
$$\mod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}_{Q,P}^4.$$

We may reduce this to

$$
\begin{aligned}
-Q_2^3 \;\equiv\;& h_1(-2Q_1 - Q_2, -Q_1 - 3aQ_2^2 - P_2 + 2P_3Q_2, -P_3)Q_1 \\
&+h_2(-2Q_1 - Q_2 - P_1, -Q_1 - 3aQ_2^2 + 2P_3Q_2, -P_3)Q_2 \\
&+h_3(-2Q_1 - Q_2 - P_1, -Q_1, -P_3)Q_2^2 + h_0(-2Q_1 - Q_2 - P_1, -Q_1 - P_2, -P_3) \\
&\ \mathrm{mod}\ \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}_{Q,P}^4.
\end{aligned}
$$

We show this equation has a contradiction. The coefficients of $P_1^{i_1}P_2^{i_2}P_3^{i_3}$ on the equation depend only on the coefficients of $q_1^{i_1}q_2^{i_2}q_3^{i_3}$ on h_0 respectively. This means that $h_0(q \circ S_a) \equiv 0$. The coefficients of Q_1^2, Q_1P_2, Q_1P_3 on the equation depend only on the coefficients of q_1, q_2, q_3 on h_1 respectively. This means that $j^1(h_1(q \circ S_a)(0) \equiv 0$. The coefficients of Q_2P_1, Q_1Q_2, Q_2P_3 on the equation depend only on the coefficients of q_1, q_1, q_3 on h_2. This means that $j^1(h_2(q \circ S_a))(0) \equiv 0$. So we need only to consider the quadratic part of h_1, h_2 and the linear part of h_3. The coefficients of $Q_2P_1^2, Q_2^2P_1$ on the equation depend only on the coefficient of q_1^2 on h_2 and the coefficient of q_1 on h_3 respectively. This means that their coefficients are all equal to 0. Therefore the coefficient of Q_2^3 on the right hand side of the equation is 0. This contradicts the equation. So we have that $j^2(\frac{dS_a}{da})(0)$ is not included in $T_z([z]_L)$.

We also prove the case $B_{2,2,3}^{+,+}$: We consider the reticular Lagrangian maps $\pi \circ i_a$ with the generating families $F_a(x_1, x_2, q_1, q_2, q_3) = (x_1 + x_2)^2 + ax_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$. Then the function germs $G_a(Q_1, Q_2, Q_3, q_1, q_2, q_3) = (Q_1 + Q_2)^2 + aQ_2^3 + q_1Q_1 + q_2Q_2 + q_3Q_2^2 + q_3Q_3$ are the generating functions of the canonical relations P_{S_a} and $i_a = S_a|_{\mathbb{L}}$. Then S_a have the forms:

$$S_a(Q, P) = (-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2 + 3aQ_2^2 + P_2 - 2P_3Q_2), -P_3, Q_1, Q_2, Q_2^2 + Q_3).$$

We have that $\frac{dS_a}{da} = (0, -3Q_2^2, 0, 0, 0, 0) = X_f \circ S_a$ for $f = -p_2^3$. Then we consider the following equation:

$$
\begin{aligned}
f \circ S_a = -Q_2^3 \;\equiv\;& h_1(q \circ S_a)Q_1 + h_2(q \circ S_a)Q_2 + h_3(q \circ S_a)(Q_2^2 + Q_3) + h_0(q \circ S_a) \\
&\ \mathrm{mod}\ \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}_{Q,P}^4,
\end{aligned}
$$

where $h_1, h_2, h_3 \in \mathfrak{M}(Q), h_0 \in \mathfrak{M}^2(Q)$. We may reduce this to

$$
\begin{aligned}
-Q_2^3 \;\equiv\;& h_1(-(2Q_1 + 2Q_2), -(2Q_1 + 2Q_2 + 3Q_2^2 + P_2 - 2Q_2P_3), -P_3)Q_1 \\
&+h_2(-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2 + 3Q_2^2 - 2Q_2P_3), -P_3)Q_2 \\
&+h_3(-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2), -P_3)Q_2^2 \\
&+h_0(-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2 + 3aQ_2^2 + P_2 - 2Q_2P_3), -P_3) \\
&\ \mathrm{mod}\ \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}_{Q,P}^4.
\end{aligned}
$$

By the same reason in the case $B_{2,3}^{+,+}$, we have that $h_0(q \circ S_a) \equiv 0$. By the consideration of the coefficients of Q_1^2, Q_1P_2, Q_1P_3 and Q_2P_1, Q_2^2, Q_2P_3 on the equation, we have that $j^1(h_1(q \circ S_a)Q_1)(0) \equiv j^1(h_2(q \circ S_a)Q_2)(0) \equiv 0$. The coefficients of $Q_1P_2^2, Q_1P_3^2, Q_1P_2P_3$ on the equation depend only on the coefficients of q_2^2, q_3^2, q_2q_3 on h_1. This means that they are all

equal to 0. The coefficients of $Q_1^2P_2, Q_1^2P_3, Q_1^3$ depend only on the coefficients of q_1q_2, q_1q_3, q_1^2 on h_1. This means that they are all equal to 0. We have that $j^2(h_1(q\circ S_a)Q_1)(0)\equiv 0$.

The coefficients of $Q_2P_1^2, Q_2P_3^2, Q_2P_1P_3$ depend only on the coefficients of q_1^2, q_3^2, q_1q_3 on h_2 and they are all equal to 0. We write $h_2 = q_2(bq_1+cq_2+dq_3), h_3 = eq_1+fq_2+gq_3$. We calculate the coefficients of $Q_1^2Q_2, Q_1Q_2^2, Q_2^2P_1, Q_1Q_2P_3, Q_2^2P_3$, then we have that $-2b-2c = -8(-2b-2c)+2e(-2-2f) = 4b-2e = d = 4d-2eg = 0$. This is solved that $b=c=d=e=0$ or $b=\frac{e}{2}, c=-\frac{e}{2}, d=0, f=-1, g=0$. This means that the coefficient of Q_2^3 on the right hand side of the equation is $4b+4c-2e-2ef=0$. This contradicts the equation.

We also prove the case $C_{3,2}^{+,+}$: We consider the reticular Lagrangian maps $\pi\circ i_a$ with the generating families $F_a(y,x_1,x_2,q_1,q_2,q_3) = y^3+x_1y+x_2y+ax_2^2+ax_2^3+q_1y+q_2x_1+q_3x_2$. Then the function germs $G_a(y,Q_1,Q_2,Q_3,q_1,q_2,q_3) = y^3+Q_1y+Q_2y+aQ_2^2+q_1y+q_2Q_1+q_3Q_2+yQ_3$ are the generating families of the canonical relations P_{S_a} and $i_a = S_a|_{\mathbb{L}}$. Then S_a have the forms:

$$S_a(Q,P) = (-(3P_3^2+Q_1+Q_2+Q_3), P_3-P_1, P_3-2aQ_2-P_2, -P_3, Q_1, Q_2).$$

We have that $\frac{dS_a}{da} = (0,0,-2Q_2,0,0,0) = X_f\circ S_a$ for $f=-p_3^2$. Then we consider the following equation:

$$\begin{aligned} f\circ S_a = -Q_2^2 \;\equiv\;& h_1(q\circ S_a)(-P_3)+h_2(q\circ S_a)Q_1+h_3(q\circ S_a)Q_2+h_0(q\circ S_a) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}}+\mathfrak{M}_{Q,P}\langle Q_3\rangle+\mathfrak{M}_{Q,P}^3. \end{aligned}$$

We may reduce this to

$$\begin{aligned} -Q_2^2 \;\equiv\;& h_1(-(Q_1+Q_2), P_3-P_1, P_3-2aQ_2-P_2)(-P_3) \\ & +h_2(-(Q_1+Q_2), P_3, P_3-2aQ_2-P_2)Q_1 \\ & +h_3(-(Q_1+Q_2), P_3-P_1, P_3-2aQ_2)Q_2 \\ & +h_0(-(Q_1+Q_2), P_3-P_1, P_3-2aQ_2-P_2) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}}+\mathfrak{M}_{Q,P}\langle Q_3\rangle+\mathfrak{M}_{Q,P}^3. \end{aligned}$$

Since the coefficients of $P_1^{i_2}P_2^{i_3}$ on the equation depend only on the coefficients of $q_2^{i_2}q_3^{i_3}$ on h_0, it follows that they are all equal to 0. Since the coefficients of P_1P_3, P_2P_3 depend only on the coefficients of q_2, q_3 on h_1, it follows that they are all equal to 0.

Therefore we may set $h_1 = bq_1,\ h_2 = cq_1+dq_2+eq_3,\ h_3 = fq_1+gq_2+hq_3,\ h_0 = q_1(iq_1+jq_2+hq_3)$. By the calculation of the equation, we have that the coefficient of Q_2^2 on the right hand side of the equation is 0. This contradicts the equation.

Lemma 3.4 *Let $\pi\circ i : (\mathbb{L},0)\to(T^*\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a reticular Lagrangian map, S be an extension of i. Suppose that the caustic-equivalence class $[j_0^{n+2}S(0)]_c$ be an immersed manifold of $S^{n+2}(n)$. If a generating family of $\pi\circ i$ is reticular $\mathcal{P}$-$\mathcal{C}$-stable and $j_0^{n+2}S$ is transversal to $[j_0^{n+2}S(0)]_c$ at 0, then $\pi\circ i$ is caustic stable.*

This is proved by the analogous method of [**?**, Theorem 6.6 (t)&(is)⇒(s)]. By this lemma, we have that the caustic-stability of reticular Lagrangian maps is reduced to the transversality of finite dimensional jets of extensions of their reticular Lagrangian embeddings.

4 Weak Caustic-equivalence

There exist modalities in the classification list of Section 2. This means that the caustic-equivalence is still too strong for a generic classification of caustics on a corner. In order to obtain the generic classification, we need to admit the following equivalence relation:

We say that two function germs in $\mathfrak{M}(r;k+n)^2$ are *weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent* if they are generating families of weakly caustic-equivalent reticular Lagrangian maps. We define the *stable* weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalence by the ordinary way.

We say that a reticular Lagrangian map $\pi \circ i$ is *weakly caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of i and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of S, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at x_0 and $\pi \circ i$ are weakly caustic-equivalent for some $x_0 = (0, \ldots, 0, p^0_{r+1}, \ldots, p^0_n) \in U$.

We say that a function germ $F(x,y,u) \in \mathfrak{M}(r;k+n)$ is *weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ in C^∞-topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is weak reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to F for some $(0, y_0, q_0) \in U$.

We here investigate the reticular $\mathcal{C}$-equivalence classes $B^{+,+,2}_{2,2,a}$ of function germs. The same methods are valid for the classes $B^{\pm,+,1}_{2,2,a}$, $B^{\pm,+,2}_{2,2,a}$, $B^{\pm,-}_{2,2,a}$. So we prove only to the classes $B^{+,+,2}_{2,2,a}$.

We consider the reticular Lagrangian maps $\pi \circ i_a : (\mathbb{L}, 0) \to (T^*\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with the generating families $F_a(x_1, x_2, q_1, q_2) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2$ $(a > \frac{1}{4})$. We give the caustic of $\pi \circ i_a$ and $\pi \circ i_b$ for $\frac{1}{4} < a < b$. In these figures $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are in the

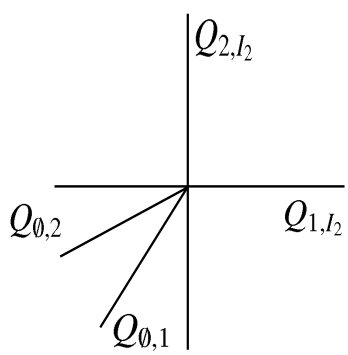

Figure 3: the caustics of $\pi \circ i_a$

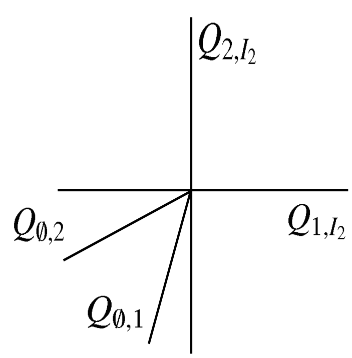

Figure 4: the caustics of $\pi \circ i_b$

same positions. Suppose that there exists a diffeomorphism germ g on $(\mathbb{R}^2, 0)$ such that $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are invariant under g. Then g can not map $Q_{\emptyset,1}$ from one to the other. This implies that caustic-equivalence is too strong for generic classifications. But these caustic are equivalent under the weak caustic-equivalence. This implies that the reticular Lagrangian map $\pi \circ i_a$ is weakly caustic equivalent to $\pi \circ i_1$ for any $a > \frac{1}{4}$ and hence F_a is weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to F_1. We remark that a homeomorphism germ g_a, which gives the weak caustic-equivalence of $\pi \circ i_1$ and $\pi \circ i_a$, may be chosen to be smooth outside

0 and depends smoothly on a. This means that the weak caustic-equivalence relation is naturally extended for the (caustic) stable reticular Lagrangian maps with the generating families $F'_a(x_1,x_2,q_1,q_2,q_3) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$ and F'_a is weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to $F'(x_1,x_2,q_1,q_2,q_3) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$. The figure of the corresponding caustic is given in [?, p.602 $B_{2,2}^{+,+,\overset{<}{\alpha}}$]. We also remark that the functions $x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ and $x_1^2 + x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$ are not weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent because $Q_{\emptyset,1}$ and $Q_{\emptyset,1}$ of their caustics are in the opposite positions to each other.

Therefore we have that the function germ $f_a(x) = x_1^2 + x_1x_2 + ax_2^2 (a > \frac{1}{4})$ are all weak reticular $\mathcal{C}$-equivalent. Since $\frac{df_a}{da} = x^2$ is not included in $\langle x\frac{\partial f_a}{\partial x}\rangle_{\mathcal{E}(x)}$, it follows that the l-jets of the weak reticular $\mathcal{C}$-equivalence class of f_a consists an immersed manifold of $J^l(2,1)$ for $l \geq 2$.

We classify function germs in $\mathfrak{M}(2;k)^2$ with respect to the weak reticular $\mathcal{C}$-equivalence with the codimension≤ 4. Then we have the following list:

k	Normal form	codim	Notation
0	$x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2$	2	$B_{2,2}^{\pm,+,1}$
	$x_1^2 \pm x_1x_2 + x_2^2$	2	$B_{2,2}^{\pm,+,2}$
	$x_1^2 \pm x_1x_2 - x_2^2$	2	$B_{2,2}^{\pm,-}$
	$x_1^2 \pm x_2^2$	3	$B_{2,2}^{\pm,0}$
	$(x_1 \pm x_2)^2 \pm x_2^3$	3	$B_{2,2,3}^{\pm,\pm}$
	$x_1^2 \pm x_1x_2 \pm x_2^3$	3	$B_{2,3}^{\pm,\pm}$
	$x_1^2 \pm x_1x_2^2 \pm x_2^3$	4	$B_{2,3'}^{\pm,\pm}$
	$x_1^3 \pm x_1^2x_2 \pm x_2^2$	4	$B_{3,2'}^{\pm,\pm}$
	$x_1^3 \pm x_1x_2 \pm x_2^2$	3	$B_{3,2}^{\pm,\pm}$
1	$\pm y^3 + x_1y \pm x_2y + x_2^2$	3	$C_{3,2}^{\pm,\pm}$
	$\pm y^3 + x_1y \pm x_2y^2 + x_2^2$	4	$C_{3,2,1}^{\pm,\pm}$
	$\pm y^3 + x_2y \pm x_1y^2 + x_1^2$	4	$C_{3,2,2}^{\pm,\pm}$

Proposition 4.1 *Let $\pi \circ i_a : (\mathbb{L},0) \to (T^*\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be the reticular Lagrangian map with the generating family $x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2$. Let $S_a \in S(T^*\mathbb{R}^2,0)$ be extensions of i_a. Then the weak caustic-equivalence class*

$$[j^lS_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^lS_a(0)]_c$$

is an immersed manifold in $S^l(2)$ for $l \geq 1$.

Proof. The function germ $G_a(Q_1,Q_2,q_1,q_2) = Q_1^2 + Q_1Q_2 + aQ_2^2 + q_1Q_1 + q_2Q_2$ is a generating function of the canonical relation P_{S_a} and we have that

$$S_a(Q,P) = (-(2Q_1 + Q_2 + P_1), -(Q_1 + 2aQ_2 + P_2), Q_1, Q_2).$$

This means that $\frac{dS_a}{da} = (0, -2Q_2, 0, 0) = X_f \circ S_a$ for $f = -p_2^2$. Suppose that $j^1(\frac{dS_a}{da})(0)$ is included in $T_z(rLa^1(2)\cdot z)$. Then there exist $h_1, h_2 \in \mathfrak{M}_{Q,P}$ and $h_0 \in \mathfrak{M}_{Q,P}^2$ such that

$$-Q_2^2 \equiv h_1(q\circ S_a)Q_1 + h_2(q\circ S_a)Q_2 + h_0(q\circ S_a) \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}^3.$$

We need only to consider the linear parts of h_1, h_2 and the quadratic part of h_0. The coefficients of P_1^2, P_2^2, P_1P_2 depend only on the coefficients of Q_1^2, Q_2^2, Q_1Q_2 on h_0 respectively. This means that $h_0 \equiv 0$. We set $h_1 = bq_1 + cq_2, h_2 = dq_1 + eq_2$ and calculate the coefficients of $Q_1^2, Q_1Q_2, Q_1P_2, Q_2P_1$ in the equation. Then we have that $-2b-c=0, -b-2d-e-2ca = 0, c=0, d=0$. This means that $e=0$. Then we have that the coefficient of Q_2^2 of the right hand side of the equation is equivalent to $-d-ae=0$. This contradicts the equation. ■

We consider the (caustic) stable reticular Lagrangian map $\pi \circ i_a : (\mathbb{L}, 0) \to (T^*\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ with the generating family $x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$ and take an extension $S'_a \in S(T^*\mathbb{R}^2, 0)$ of i_a, then we have by the analogous method that:

Corollary 4.2 *Let S'_a be as above. Then the weak caustic-equivalence class*

$$[j^l S'_1(0)]_w := \bigcup_{a > \frac{1}{4}} [j^l S'_a(0)]_c$$

is an immersed manifold in $S^l(3)$ for $l \geq 1$.

Since the caustic of $\pi \circ i_a$ is given by the restrictions of $\pi \circ i_a$ to $L_\sigma^0 \cap L_\tau^0$ for $\sigma \neq \tau$ in this case, it follows that the caustic is determined by the linear part of i_a. This means that $\pi \circ i_a$ is 1-determined with respect to the weak caustic-equivalence (cf., Definition 3.1).

Theorem 4.3 *The function germ $F(x_1, x_2, q_1, q_2) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ is a weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$*

Proof. We define $F' \in \mathfrak{M}(2;3)^2$ by $F'(x_1, x_2, q_1, q_2, q_3) = F(x_1, x_2, q_1, q_2) + q_3x_2^2$ Then F' is a reticular $\mathcal{P}$-$\mathcal{R}^+$-stable unfolding of f. It follows that for any neighborhood U' of 0 in $\mathbb{R}^5$ and any representative $\tilde{F}' \in C^\infty(U, \mathbb{R})$, there exists a neighborhood $N_{\tilde{F}'}$ such that for any $\tilde{G}' \in N_{\tilde{F}'}$ the function germ $\tilde{G}'|_{\mathbb{H}^2 \times \mathbb{R}^3}$ at p'_0 is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to F' for some $p'_0 = (0, 0, q_1^0, q_2^0, q_3^0) \in U'$.

Let a neighborhood U of 0 in $\mathbb{R}^4$ and a representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ be given. We set the open interval $I = (-0.5, 0.5)$ and set $U' = U \times I$. Then there exists $N_{\tilde{F}'}$ for which the above condition holds. We can choose a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any $\tilde{G} \in N_{\tilde{F}}$ the function $\tilde{G} + q_3x_2^2 \in N_{\tilde{F}'}$. Let a function $\tilde{G} \in N_{\tilde{F}}$ be given. Then the function germ $G' = (\tilde{G} + q_3x_2^2)|_{\mathbb{H}^2 \times \mathbb{R}^3}$ at p'_0 is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to F' for some $p'_0 = (0, 0, q_1^0, q_2^0, q_3^0) \in U'$. We define $G \in \mathfrak{M}(2;2)^2$ by $\tilde{G}|_{\mathbb{H}^2 \times \mathbb{R}^2}$ at $p_0 = (0, 0, q_1^0, q_2^0) \in U$. Then it holds that $G'(x, q) = G(x, q_1, q_2) + (q_3 + q_3^0)x_2^2$, and $G'|_{q=0} = G(x, 0) + q_3^0x_2^2$ is reticular $\mathcal{R}$-equivalent to f. Let (Φ, a) be the reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalence from G' to F'. We write $\Phi(x, q) = (x\phi_1(x, q), \phi_1^2(q), \phi_2^2(q), \phi_3^2(q))$. By shrinking U if necessary, we may assume that the map germ

$$(q_1, q_2) \mapsto (\phi_1^2(q_1, q_2, 0), \phi_2^2(q_1, q_2, 0)) \text{ on } (\mathbb{R}^2, 0)$$

is a diffeomorphism germ. Then F is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to $G_1 \in \mathfrak{M}(2;2)^2$ given by $G_1(x, q) = G(x_1, x_2, q_1, q_2) + (\phi_3^2(q_1, q_2, 0) + q_3^0)x_2^2$. It follows that the reticular Lagrangian maps defined by F and G_1 are Lagrangian equivalent. We have that

$$j^2(G + q_3^0x_2^2)(0) = j^2G_1(0), \quad q_3^0 > -0.5.$$

This means that the caustic of G_1 is weakly caustic-equivalent to the caustic of G because the reticular Lagrangian maps of G_1 and F are the same weak caustic-equivalence class that

is 1-determined under the weak caustic-equivalence. This means that F and G are weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent. Therefore F is weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable. ■

By the above consideration, we have that: For each singularity $B_{2,2}^{\pm,+,1}, B_{2,2}^{\pm,+,2}, B_{2,2}^{\pm,-}$, if we take the symplectic diffeomorphism germ $S_a(S'_a)$ as the above method, then the weak caustic-equivalence class $[j^l S_a(0)]_w([j^l S'_a(0)]_w)$ is one class and immersed manifold in $S^l(2)(S^l(3))$ for $l \geq 1$ respectively.

We now start to prove the main theorem: We choose the weakly caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ for

$$X = B_{2,2}^{\pm,+,1}, B_{2,2}^{\pm,+,2}, B_{2,2}^{\pm,-}. \tag{2}$$

We also choose the caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L}, 0) \to (T^*\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ for

$$X = B_{2,2}^{\pm,0}, B_{2,2,3}^{\pm,\pm}, B_{2,3}^{\pm,\pm}, B_{3,2}^{\pm,\pm}, B_{2,3'}^{\pm,\pm}, B_{3,2'}^{\pm,\pm}, C_{2,3}^{\pm,\pm}, C_{3,2,1}^{\pm,\pm}, C_{3,2,2}^{\pm,\pm}. \tag{3}$$

Then other reticular Lagrangian maps are not caustic-stable since other singularities have reticular $\mathcal{C}$-codimension > 4. We choose extensions $S_X \in S(T^*\mathbb{R}^n, 0)$ of i_X for all X. We define that

$$O'_1 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } [j^{n+2}S_X(0)]_w \text{ for all } X \text{ in (2)}\},$$

$$O'_2 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } [j^{n+2}S_X(0)]_c \text{ for all } X \text{ in (3)}\},$$

where $j_0^l\tilde{S}(x) = j^l\tilde{S}_x(0)$. Then O'_1 and O'_2 are residual sets. We set

$$Y = \{j^{n+2}S(0) \in S^{n+2}(n) \mid \text{the codimension of } [j^{n+2}S(0)]_L > 10\}.$$

Then Y is an algebraic set in $S^{n+2}(n)$ by [**?**, Theorem 6.6 (a')]. Therefore we can define that

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } Y\}.$$

For any $S \in S(T^*\mathbb{R}^n, 0)$ with $j^{n+2}S(0)$ and any generating family F of $\pi \circ S|_{\mathbb{L}}$, the function germ $F|_{q=0}$ has the reticular $\mathcal{R}^+$-codimension > 5. This means that $F|_{q=0}$ has the reticular C-codimension > 4. It follows that $j^{n+2}S(0)$ does not belong to the above equivalence classes. Then Y has codimension > 8. Then we have that

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S}(U) \cap Y = \emptyset\}.$$

We define $O = O'_1 \cap O'_2 \cap O''$. Since all $\pi \circ i_X$ for X in (2) are weak caustic 1-determined, and all $\pi \circ i_X$ in (3) are caustic 5-determined by Lemma 3.2. Then O has the required condition. ■

$B^{*;*}_{2,2}$ in $2D$

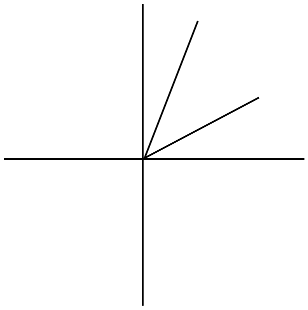

Figure 5: $B^{+;+,1}_{2,2}$, $B^{+;+,2}_{2,2}$

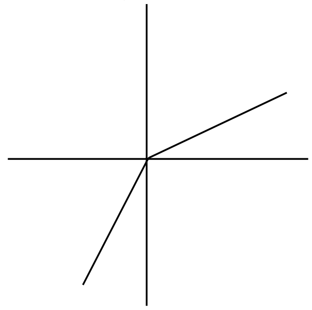

Figure 6: $B^{-;+,1}_{2,2}$, $B^{-;+,2}_{2,2}$

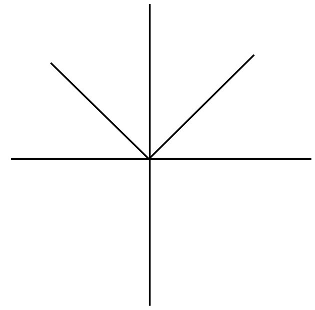

Figure 7: $B^{+;-}_{2,2}$, $B^{-;-}_{2,2}$

$B^{*;*}_{2,2}$ in $3D$

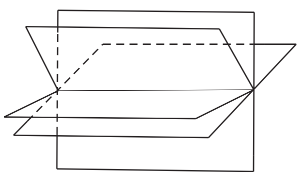

Figure 8: $B^{+;+,1}_{2,2}$, $B^{+;+,2}_{2,2}$

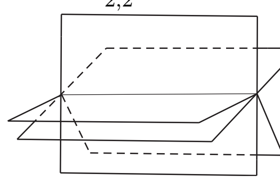

Figure 9: $B^{-;+,1}_{2,2}$, $B^{-;+,2}_{2,2}$

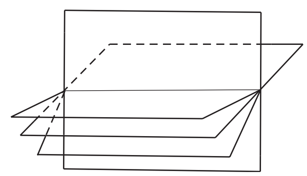

Figure 10: $B^{+;-}_{2,2}$, $B^{-;-}_{2,2}$

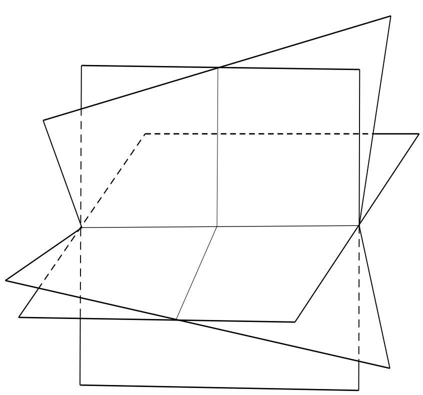

Figure 11: $B^{+;0}_{2,2}$

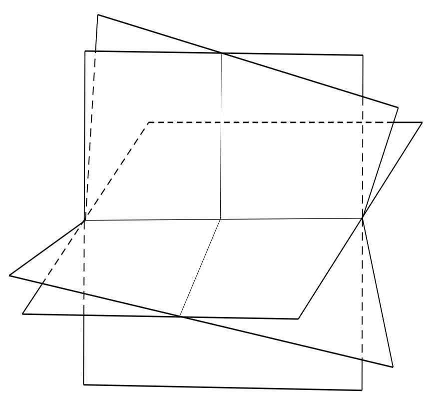

Figure 12: $B^{-;0}_{2,2}$

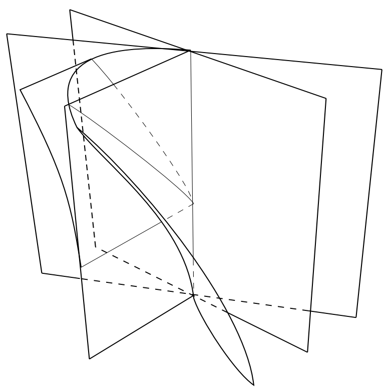

Figure 13: $B^{+,+}_{2,2,3}$

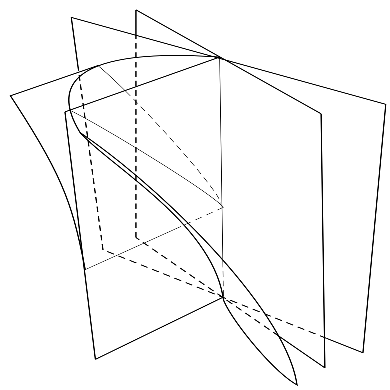

Figure 14: $B^{+,-}_{2,2,3}$

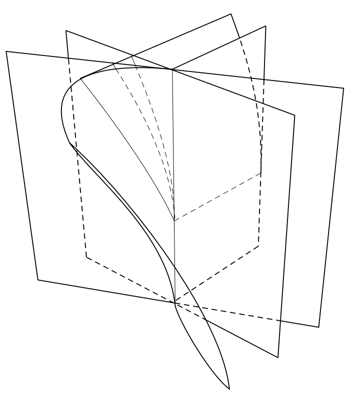

Figure 15: $B^{-,+}_{2,2,3}$

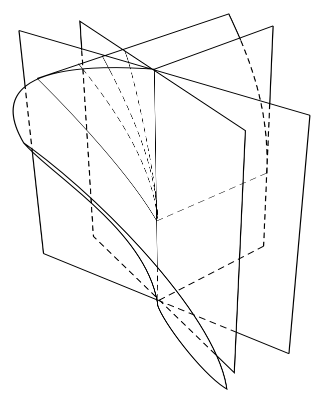

Figure 16: $B^{-,-}_{2,2,3}$

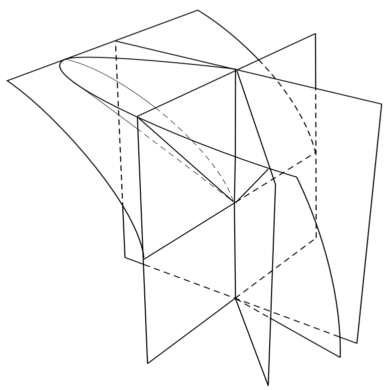

Figure 17: $B^{+;+}_{2,3}$

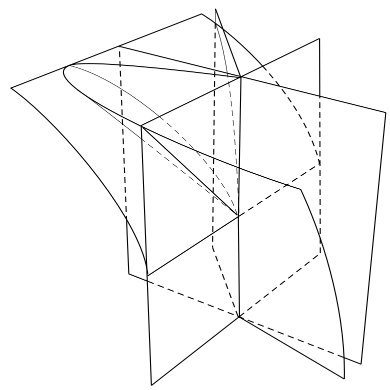

Figure 18: $B^{+;-}_{2,3}$

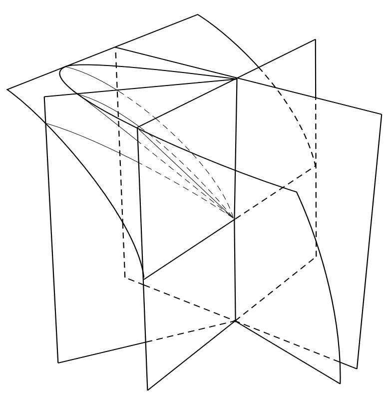

Figure 19: $B^{-;+}_{2,3}$

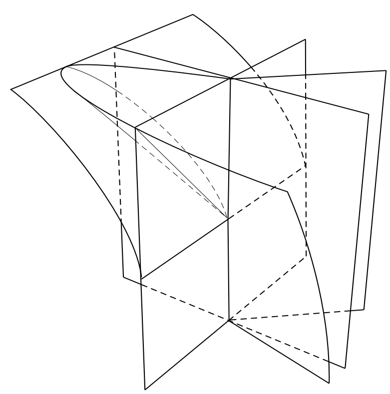

Figure 20: $B^{-;-}_{2,3}$

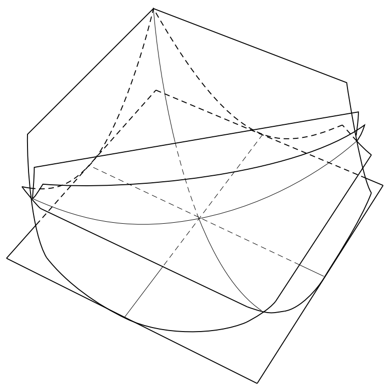

Figure 21: $C_{3,2}^{+,+}$

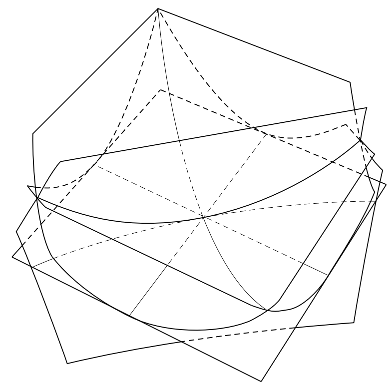

Figure 22: $C_{3,2}^{+,-}$

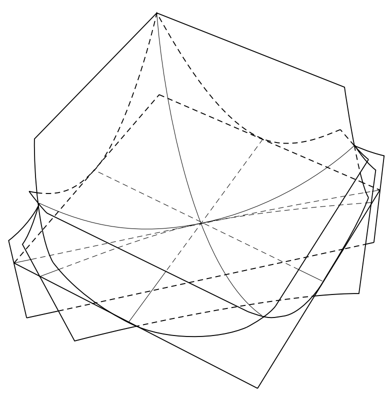

Figure 23: $C_{3,2}^{-,+}$

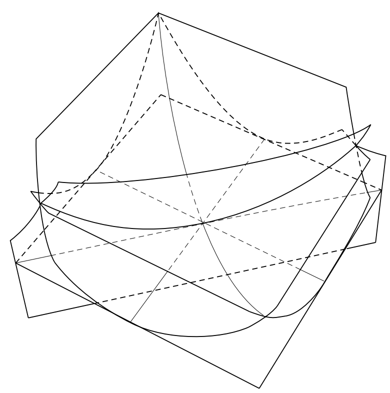

Figure 24: $C_{3,2}^{-,-}$